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Decomposition of Complete 3-uniform Hypergraphs K_n^3 into Cycles for $n = 7, 10$

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Abstract

A k-uniform hypergraph H is a pair (V, ε), where V is a set $V = \{v_1, v_2, \ldots, v_n\}$ of n vertices and ε is a family of k-subset of V called hyperedges. A cycle of length l of H is a sequence $(v_1, e_1, \ldots, v_l, e_l, v_1)$, where $v_1, v_2, \ldots, v_l$ are distinct vertices, and $e_1, e_2, \ldots, e_l$ are k-edges of H and $v_i, v_{i+1} \in e_i, 1 \le i \le l$, where addition on the subscripts is modulo n, $e_i \ne e_j$ for $i \ne j$. We consider the problem of constructing such decompositions for complete uniform hypergraphs. In this paper we apply design theory to give the decomposition of complete 3-uniform hypergraph K_n^3 into cycles for $n = 7, 10$.

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1 Introduction

A decomposition of a graph $G = (V, E)$ is a partition of the edge-set E; a Hamiltonian decomposition of G is a decomposition into Hamiltonian cycles. The problem of constructing Hamiltonian decompositions is a long-standing and well-studied one in graph theory; in particular, for the complete graph K_n, it was solved in the 1890s by Walecki [1]. Walecki showed that K_n has a Hamiltonian decomposition if and only if n is odd, while if n is even K_n has a decomposition into Hamiltonian cycles and a perfect matching. As with many problems in graph theory, it seems natural to attempt a generalisation to hypergraphs. Indeed, the notion of Hamiltonicity was first generalised to uniform hypergraphs by Berge in his 1970 book [2]. His definition of a Hamiltonian cycle in a hypergraph $H = (V, E)$ is a sequence $(v_0, e_1, v_1, e_2, \cdots, v_{n-1}, e_n, v_0)$, where $\{v_0, \cdots, v_n\} = V$, and $e_1, \cdots, e_n$ are distinct elements of E, such that the hyperedge e_i contains both v_{i+1} and $v_i (modulo\, n)$. The study of decompositions of complete 3-uniform hypergraphs into cycles of this type was begun by Bermond et al in the 1970s [3] and was completed by Verrall in 1994 [4]. A k-uniform hypergraph H is a pair (V, ε), where $V = \{v_1 v_2, \ldots, v_n\}$ is a set of n vertices and ε is a family of k-subset of V called hyperedges. If ε consists of all k-subsets of V, then H is a complete k-uniform hypergraph on n vertices and is denoted by K_n^k. At the same time we may refer a vertex $v_i \in V$ to v_{i+n}. A cycle of length l of H is a sequence of the form

$$(v_1, e_1, v_2, e_2, \ldots, v_l, e_l, v_1),$$

where $v_1, v_2, \ldots, v_l$ are distinct vertices, and $e_1, e_2, \ldots, e_l$ are k-edges of H, satisfying

(i) $v_i,\ v_{i+1} \in e_i$, $1 \le i \le l$, where addition on the subscripts is modulo n, and

(ii) $e_i \ne e_j$ for $i \ne j$. This cycle is known as a Berge cycle, having been introduced by Berge in [1]. A cycle of length l decomposition of H is a partition of the hyperedges of H into cycles of length l.

The set of cycles of length l of complete 3-uniform hypergraph K_n^3, say C_1, $\ldots$, C_m, is called cycles of length l decomposition if $\bigcup_{i=1}^m \varepsilon(C_i) = \varepsilon(K_n^3)$ and $\varepsilon(C_i) \cap \varepsilon(C_j) = \emptyset$ for $i \ne j$. In this paper, we apply design theory to give decomposition of complete 3-uniform hypergraph.

2 Main results

We notice a Hamiltonian cycle in K_n^k is an example of a 1-(n,k,k) design; clearly, each vertex (i.e. point) lies in exactly k edges. Therefore a Hamiltonian decomposition of K_n^k is, in the language of design theory, a large set of 1-(n,k,k) designs. So one may ask what known results in the design theory literature may be of use to us here.

Definition 2.1. *Let v, k and λ be integers such that $v \geq k \geq 2$ and $\lambda \geq 1$. Let X be a finite set of elements, called points, and let $\mathcal{B}$ be a finite collection of subsets of X, called blocks. The pairs $(X, \mathcal{B})$ is called a (v, k, λ) balanced incomplete block design or, simply, a $(v, k, \lambda)-BIBD$, if the following conditions hold:*

(i) $|X| = v$.

(ii) $|B| = k$ for all $B \in \mathcal{B}$.

(iii) Every pairs of distinct points is contained in exactly λ blocks.

The set $\{v, k, \lambda\}$ is called the set of parameters of the $BIBD$ $(X, \mathcal{B})$. We also use notation $\mathcal{D} = (X, \mathcal{B})$.

Definition 2.2. *Let v, k and λ be integers such that $v \geq k \geq 2$ and $\lambda \geq 1$. Let X be a finite set of elements, called points, and let $\mathcal{B}$ be a finite collection of subsets of X, called blocks. The pairs $(X, \mathcal{B})$ is called a $t-(v, k, \lambda)$ design or, simply, a $t-$design, if the following conditions hold:*

(i) $|X| = v$.

(ii) $|B| = k$ for all $B \in \mathcal{B}$.

(iii) Every subset of t distinct points is contained in exactly λ blocks. The set $\{t; v, k, \lambda\}$ is called the set of parameters of the $t-design$ $(X, \mathcal{B})$.

We have a 3-(7,5,1) design as followed:

{1,2,3,5,6}
{2,3,4,6,7}
{1,3,4,5,7}
{1,3,5,6,7}
{2,3,4,6,7}

{1,2,4,5,6}
{1,2,4,5,7}

The sets above are happen to be a decomposition of complete 3-uniform hypergraph K_7^3, which has seven cycles of length 5. We can easily see from the above Definition 2.2 when $k = \lambda, t = 1$, a $1-(v,k,k)$ design happen to be a Hamiltonian cycle of a k-uniform hypergraph.

For the simpleness, we omit the set sign $\{\cdots\}$.

Complete 3-uniform hypergraph K_7^3 can decompose into 5 Hamiltonin cycles of length 7 or 7 cycles of length 5.

Let vertex set be $\{1, 2, 3, 4, 5, 6, 7\}$, we have five $1-(7, 3, 3)$ designs as followed

013,135,356,456,246,024,012
015,125,235,236,346,046,014
023,234,345,145,156,016,026
034,134,146,145,126,025,035
045,245,124,123,136,036,056

All the 35 3-subsets divided into 5 lines, every 3-subset of any line is a hyperedge of hypergraph, every line happen to be a Hamiltonian cycle, 5 lines are the Hamiltonian decomposition of complete 3-uniform hypergraph K_7^3. We also can arrange 35 3-subsets as followed:

123,235,356,156,126
234,346,046,026,023
345,045,015,013,134
135,035,056,016,136
034,036,236,246,024
456,146,124,125,256
145,245,025,012,014

Every line happen to be a cycle of length 5, so we have a decomposition of complete 3-uniform hypergraph K_7^3 into cycles length 5.

Definition 2.3. *t-wise* Γ *balance design, is a pair* $(X, \mathcal{B})$*,* Γ *is a set consisting of t-uniform hypergraphs,* Ω *is set consisting of complete t-uniform hypergraph,* X *is a finite set with* v *vertices,* $\mathcal{B}$ *is a hypergraph on the subsets of* X*,* *such that for any block $B \in \mathcal{B}$, B is isomorphic one of Γ, and every t-subset is included an only block, denoted by $S(t,\Gamma,v)$. If replacing Γ by Ω, then we use $S(t,K,v)$ express $S(t,\Omega,v)$, where K is an positive integer set. The number of element from B come from K.*

Base on this definition, we could get a Hamiltonian decomposition of complete 3-uniform hypergraph K_{10}^3 as followed:

012,345,678,123,234,456,567,789,089,019
029,249,479,457,357,356,168,136,018,028
013,035,058,568,468,467,247,279,129,139
124,146,169,679,579,578,358,038,023,024
235,257,027,078,068,689,469,149,134,135
346,368,138,189,179,079,057,025,245,246
034,348,389,589,159,125,127,267,067,046
145,459,049,069,026,236,238,378,178,157
256,056,015,017,137,347,349,489,289,268
367,167,126,128,248,458,045,059,039,379
478,278,237,239,359,569,156,016,014,048
036,369,269,259,258,158,147,148,047,037

This is a Hamiltonian decomposition of complete 3-uniform hypergraph K_{10}^3, every line is a Hamiltonian cycle, every line is isomorphic the others and every one is 1-(10,3,3) design of X.

Complete 3-uniform hypergraph K_{10}^3 also can decompose into 10 Hamiltonian cycles and 4 cycles of length 5 as followed:

012,345,678,123,234,456,567,789,089,019
036,369,269,259,258,158,147,148,047,037
078,578,568,356,346,134,124,129,029,079
189,689,679,467,457,245,235,023,013,018
056,156,126,127,278,378,348,349,049,059
167,267,237,238,389,489,459,045,015,016
038,138,168,169,469,479,247,257,025,035
149,249,279,027,057,058,358,368,136,146
067,367,236,239,289,589,458,145,014,017
178,478,347,034,039,069,569,256,125,128

024,246,468,068,028
135,357,579,179,139
048,248,268,026,046
159,359,379,137,157

Every one of the first 10 lines is a Hamiltonian cycle, every one of last 4 lines is a cycle of length 5. Then we get a decomposition of complete 3-uniform hypergraph K_{10}^3 into Hamiltonian cycles and cycles of length 5 and 10, which is corresponding to the Definition 2.3 for $K = \{5, 10\}$.

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We also briefly consider decompositions of 3-uniform hypergraphs into (not necessarily Hamiltonian) cycles and comment on a possible analogue of Alspachâ€™s conjecture for cycle decompositions of the ordinary complete graph. View. Show abstract. Hamiltonian decomposition of complete bipartite r -hypergraphs. Article. Oct 2001.Â The problem of finding a Hamilton decomposition of the complete 3-uniform hypergraph K3n has been solved for n â‰¡ 2(mod 3) and n â‰¡ 4(mod 6) (Bermond, 1978). We find here a Hamilton decomposition of K3n, n â‰¡ 1(mod 6), and a Hamilton decomposition of the complete 3-uniform hypergraph minus a 1-factor, K3nâˆ'I, nâ‰¡0(mod 3), and thereby complete the problem. View. Show abstract. Decompositions of 3-uniform hypergraph K_v^{(3)} into hypergraph K_4^{(3)}+e. Item Preview. remove-circle.Â In this paper it is established that a decomposition of a 3-uniform hypergraph K_v^{(3)} into a special kind of hypergraph K_4^{(3)}+e exists if and only if v\equiv 0,1,2 (mod 5) and v\geq 7. Addeddate. 2013-07-19 23:29:50. A Decomposition of Complete Bipartite 4-Uniform Hypergraphs K. 54,5. into Loose Cycles of Length 5, 10.Â all k-subsets of V , then H is a complete k-uniform hypergraph on n vertices and is denoted by Knk. At the same time we may refer a vertex vi âˆˆ V to vi+n. A cycle of length l of H is a sequence of the form. (v1, e1, v2, e2, . . . , vl, el, v1), where v1, v2, . . . , vl are distinct vertices, and e1, e2, . . . , el are k-edges of H, satisfying. (i) vi, vi+1 âˆˆ ei, 1 â‰¤ i â‰¤ l, where addition on the subscripts is modulo n, and. (ii) ei = ej for i = j. This cycle is known as a Berge cycle, having been introduced by Berge in [2]. A cycle of length l decomposition of H is a partition. So a 2-uniform hypergraph is a graph, a 3-uniform hypergraph is a collection of unordered triples, and so on. An undirected hypergraph is also called a set system or a family of sets drawn from the universal set. Hypergraphs can be viewed as incidence structures.Â Although hypergraphs are more difficult to draw on paper than graphs, several researchers have studied methods for the visualization of hypergraphs.Â In contrast with the polynomial-time recognition of planar graphs, it is NP-complete to determine whether a hypergraph has a planar subdivision drawing,[24] but the existence of a drawing of this type may be tested efficiently when the adjacency pattern of the regions is constrained to be a path, cycle, or tree.[25]. A maximum k-uniform acyclic hypergraph of order n has size n âˆ' k + 1. An acyclic decomposition of a hypergraph H = (X , A) is a set of acyclic hypergraphs {(X , Ai)}ci=1 such that the following. conditions hold: (i) Ai âŠ† A for all i âˆˆ [c]. (ii) Ai âˆ© Aj = âˆ… for all distinct i, j âˆˆ [c]. (iii) âˆªci=1 Ai = A. The size of the acyclic decomposition is c, the number of acyclic hypergraphs in the decomposition. Definition 2.2. The arboricity of a hypergraph H , denoted arb(H), is the minimum size of an acyclic decomposition of H . 102 Y.M. Chee et al. / Discrete Applied Mathematics 160 (2012) 100â€“107.Â The purpose of this section is to determine the exact value of arb(Kn(nâˆ'3)) completely. Corollary 4.2(iii) already gives. arb(Kn(nâˆ'3)).